Non-Commutative Bayesian Expectation and its connection to Quantum Theory

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Abstract. Bayesian probability considers probabilities as degrees of plausibility that must be updated according to newly available information or evidence. In the standard application of this theory, the possible side effects of the act of measurement are not considered, and because of this the results of the updating do not depend on the ordering of two such measurements. In this work we develop the application of Bayesian probability theory to non-commutative measurements on a system. We show that the resulting formalism can be cast in an abstract way which is surprisingly close to quantum theory, together with a complex Hilbert space, linear operators representing measurements and a density operator encoding a state of knowledge.

INTRODUCTION

An essential property of quantum systems is that they are disturbed by the act of observation, a phenomenon known as the *observer effect*. This point was already highlighted by P.A.M. Dirac as the defining feature of quantum theory, and a necessary condition for a consistent definition of what constitutes a "small" system: a system affected by measurements performed on it [1]. Perhaps more interestingly, this quality of being affected by observation *is not exclusively a property of quantum systems*: it is recognized and important in several fields outside physics such as human cognition, psychology and sociology, where the act of gathering information about a system clearly influences said system. In the case of human cognition, the case has actually been made for the use of a quantum-like formalism [2, 3, 4, 5] in order to account for the effect of ordering in repeated questions.

Even outside the study of human behavior, invasive (or even destructive) measurements on any system (chemical, electrical, biological) are almost by definition non-commutative, and this fact can introduce additional sources of uncertainty into the measurements. Consider as an example the possibility of monitoring current and/or voltage of a working electrical system without disturbing its natural dynamics. If the disturbance cannot be avoided entirely, how can we compute its effects on the total uncertainty on the final measurement?

Having these examples in mind, we will define a *fragile system* as the kind of system for which an observation (measurement) produces side effects that modify its internal state in a way that is relevant for a consecutive observation of a different property. When performing this second observation on the modified internal state of the system, the result might differ from the case where the initial observation was not performed. This is in fact a generalization of Dirac's definition of what constitutes a small system. In short, fragility is not a property solely of quantum systems, some fragile systems are classical, complex systems.

It is in this sense that the possibility of a general formalism of inference prepared to deal with invasive, or, in general, non-commutative measurements, that is, with fragile systems, is suggestive, and the most suited starting point seems to be the framework of Bayesian probability, where probabilities are not objective features of the real world but instead are understood as "bookkeeping" devices for the knowledge that we currently have about a system.

In this work we develop a simple framework of non-commutative expectation in fragile systems, that is, systems highly sensitive to observation, based on Bayesian probability, and show that it can be cast in the form of a quantum-like formalism with operators expressed as matrices and a density operator containing the information we have about a system. It is important to stress here that we are not interested in recovering quantum *physics*, together with Hamiltonian operators, time evolution and the Schrödinger equation, only to show that its underlying mathematical structure can be understood as a *theory of reasoning under uncertainty* taking into account side effects. Here *time only appears implicitly* as we distinguish between an initial measurement A and a subsequent measurement B.

NON-COMMUTATIVE EXPECTATION

First, let us introduce the concepts in more detail, and some notation. Consider a system in the initial internal state s when a measurement A is performed. After "returning" the value $A(s)$ as the outcome of the measurement, the system is left in a different internal state s'. We will use primed variables to indicate that a given quantity must be evaluated after the system is modified by a measurement. After the measurement A is performed, the system goes to one of the possible states s' following a transition probability $P(s'|s,A)$ which is only dependent on the original state s and the measurement A. We recover the case of a non-fragile system, which is unaffected by measurements, with the trivial transition probability $P(s'|s,A) = \delta(s',s)$ that makes $s' = s$ for all s.

If we only consider a single measurement at a time, there is no formal difference between fragile and non-fragile systems. This is because the expected values of independent measurements of A and B in a Bayesian [6] state of knowledge I can be written, as usual, as

$$\big\langle A \big\rangle_I = \sum_s P(s|I)A(s), \qquad \big\langle B \big\rangle_I = \sum_s P(s|I)B(s), \tag{1}$$

where the altered state s' is not involved. However, there is no unique definition of the expectation of the product of two consecutive measurements of A and B for fragile systems, because this expected value depends on the ordering of the measurements. For instance, the expected value of the product $B * A$ where B is measured after A (we use the symbol $*$ to denote a non-commutative product, to be defined more precisely later) is given by

$$\big\langle B * A \big\rangle_I = \Big\langle B(s')A(s) \Big\rangle_{I,A} = \sum_{s,s'} P(s',s|I,A)B(s')A(s) = \sum_s P(s|I) \left[\sum_{s'} P(s'|s,A)B(s')\right] A(s), \tag{2}$$

while the expected value of the product where A is measured after B is

$$\big\langle A * B \big\rangle_I = \big\langle A(s')B(s) \big\rangle_{I,B} = \sum_s P(s|I) \left[\sum_{s'} P(s'|s,B)A(s')\right] B(s). \tag{3}$$

In general, then $\big\langle B * A \big\rangle_I \neq \big\langle A * B \big\rangle_I$, unless the system is not fragile, in which case $P(s'|s,A) = \delta(s',s)$ and we have

$$\big\langle B * A \big\rangle_I = \big\langle A * B \big\rangle_I = \sum_s P(s|I)A(s)B(s), \tag{4}$$

as usual. The fragility of the system not only implies that pairs of consecutive measurements do not necessarily commute, but also that joint measurements can be incompatible. For instance, if $P(s'|s,A)$ has support $\mathscr{S}_A$ and $P(s'|s,B)$ has support $\mathscr{S}_B$ such that $S_A \cap S_B =$, it will be imposible to measure A and B simultaneously.

Another way to view the non-commutative expectation in eq. (2) is to interpret the measurement of a product such as $\big\langle B * A \big\rangle_I$ as the expectation of the product $B'(s)A(s)$ where the function $B(s)$ gets "distorted" into a new function $B'(s)$, given by the expression inside square brackets in eq. (2),

$$B'(s) = \sum_{s'} B(s')P(s'|s,A), \tag{5}$$

so we can write eq. (2) as

$$\big\langle B * A \big\rangle_I = \sum_s P(s|I)B'(s)A(s) = \big\langle B'\, A \big\rangle_I. \tag{6}$$

In the same way, $\big\langle A * B \big\rangle_I = \big\langle A'\, B \big\rangle_I$ with

$$A'(s) = \sum_{s'} A(s')P(s'|s,B). \tag{7}$$

In other words, the expectations of products of consecutive measurements can always be written in the usual manner (as in eq. (1)) but with a function of s whose "shape" is dependent of the ordering of the original factors, as in general $B'(s)A(s) \neq A'(s)B(s)$. We can extend the previous notation to write the function $B'(s)A(s)$ itself as $(B * A)(s)$, thus

s	$A(s)$	$B(s)$	$A'(s)$	$B'(s)$
0	1	0	2/5	9/10
1	1	1	0	0
2	1	0	1/2	3/4
3	0	1	1/10	7/10
4	0	0	1	1
5	0	1	1/2	1/2

s	s'	$P(s'\|s,A)$
0	1	9/10
0	2	1/10
1	0	1/2
1	2	1/2
2	0	1/4
2	1	3/4
3	4	3/10
3	5	7/10
4	3	1/2
4	5	1/2
5	3	1/2
5	4	1/2

s	s'	$P(s'\|s,B)$
0	2	2/5
0	4	3/5
1	3	3/5
1	5	2/5
2	0	1/2
2	4	1/2
3	1	1/10
3	5	9/10
4	0	3/10
4	2	7/10
5	1	1/2
5	3	1/2

TABLE I. A system with 6 internal states, denoted by the labels $0,1,2,3,4$ and 5, and two measurable properties $A(s)$ and $B(s)$. The "distorted" properties $A'(s)$ and $B'(s)$ are calculated according to the transition probabilities $P(s'|s,A)$ and $P(s'|s,B)$.

removing the possible ambiguity in expressions like $\big\langle B*A\big\rangle_I$ and giving a precise mathematical definition to the non-commutative product $*$, as

$$(B*A)(s) := \sum_{s'} P(s'|s,A)B(s')A(s). \tag{8}$$

This product combines two functions of s into a new function of s which is dependent on the ordering of the factors.

A SIMPLE EXAMPLE

Consider a system with 6 possible internal states, i.e. with $s,s' \in \{0,1,2,3,4,5\}$, and two measurable properties $A(s)$ and $B(s)$ as defined in Table I. The system is clearly fragile, as shown by the transition probabilities in the same table. Using these values, we can determine for a state of knowledge I_0 in which every internal state is equally probable, i.e., $P(s|I_0) = 1/6$, the expectation values

$$\begin{aligned} \big\langle A\big\rangle_{I_0} &= 1/2, & \big\langle B\big\rangle_{I_0} &= 1/2, \\ \big\langle B*A\big\rangle_{I_0} &= 1.65, & \big\langle A*B\big\rangle_{I_0} &= 0.6. \end{aligned}$$

Here we see that B and A do not commute, and in fact have a commutator $[A,B]$ with expected value

$$\Big\langle [A,B]\Big\rangle_{I_0} = \Big\langle A*B - B*A\Big\rangle_{I_0} = \text{-}1.05.$$

A FORMULATION INDEPENDENT OF THE INTERNAL STATES

Up to this point the formalism can be, in principle, used to compute expectations taking into account the non-commutativity of measurements. However, it can be expressed in a more powerful way by avoiding references to the internal states s and s'. We will consider, without loss of generality, the expectation $\big\langle B*A\big\rangle_I$ (i.e. with A measured first, then B). It is clear that a multiple product like $(\ldots*C*B*A)$ can be reduced to a binary product by repeated application of the rules above, and that a single measurement A is equivalent to $\mathrm{I}*A$ with I the identity measurement, such that

$$\begin{aligned} P(s'|s,\mathrm{I}) &= \delta(s,s'), \\ \mathrm{I}(s) &= 1\ \forall s. \end{aligned} \tag{9}$$

Now our aim is to write $\big\langle B*A\big\rangle_I$ in a form independent of the internal variables s and s'. In order to do this we will introduce a new observable E, which is in general different from A and B, having values $E(s) \in \{\varepsilon_1,\ldots,\varepsilon_N\}$.

The choice of observable is in principle arbitrary, just as the choice of basis in a quantum mechanical calculation or a coordinate system in classical mechanics. It merely allow us to avoid referring to the internal variables and the measurements. As an illustration of this point, let us first rewrite $\big\langle A\big\rangle_I$ in terms of E, before we extend this idea to the non-commutative expectation $\big\langle B*A\big\rangle_I$. We have

$$\Big\langle A\Big\rangle_I = \sum_s P(s|I)A(s) = \sum_{i=1}^{N}\sum_s P(s,\varepsilon_i|I)A(s) = \sum_{i=1}^{N} P(\varepsilon_i|I)\left[\sum_s P(s|\varepsilon_i)A(s)\right] = \sum_{i=1}^{N} P(\varepsilon_i|I)\big\langle A\big\rangle_{\varepsilon_i} = \sum_{i=1}^{N} p_i A_i, \tag{10}$$

where in the second line we have introduced the variable ε_i through the marginalization rule. The vector components $p_i = P(\varepsilon_i|I)$ and $A_i = \big\langle A\big\rangle_{\varepsilon_i}$ can be thought of as the representations of $P(s|I)$ and $A(s)$ in the "basis" E. For clarity of notation, let us define the observable $C(s) := (B*A)(s)$. Applying the same idea as we did for A, we can write the expectation of $C = B*A$ as

$$\big\langle C\big\rangle_I = \sum_s P(s|I)\left[\sum_{s'} B(s')P(s'|s,A)\right]A(s) = \sum_{i=1}^{N}\sum_{j=1}^{N} P(\varepsilon_i,\varepsilon'_j|I)\big\langle B(s')\,A(s)\big\rangle_{\varepsilon_i,\varepsilon'_j,A} = \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij}C_{ij}, \tag{11}$$

where we have defined the $N\times N$ real matrices $\mathbb{P}$ and $\mathbb{C}$ by their matrix elements

$$p_{ij} = P(\varepsilon_i,\varepsilon'_j|I), \qquad C_{ij} = \big\langle B(s')A(s)\big\rangle_{\varepsilon_i,\varepsilon'_j,A}, \tag{12}$$

respectively. As in the example before, $\mathbb{P}$ and $\mathbb{C}$ are the representations of $P(s',s|I)$ and $(B*A)(s)$ in the "basis" E. We can write the matrix elements C_{ij} in a more explicit form, in terms of $B(s)$, $A(s)$ and $P(s'|s,A)$ as

$$\begin{aligned} C_{ij} &= \sum_s\sum_{s'} P(s,s'|\varepsilon_i,\varepsilon'_j,A)B(s')A(s) \\ &= \sum_s\sum_{s'} \frac{P(\varepsilon_i,\varepsilon'_j|s,s',A)P(s,s'|A)}{P(\varepsilon_i,\varepsilon'_j|A)}B(s')A(s) \\ &= \frac{\sum_s\sum_{s'} P(s|\varepsilon_i)P(s'|s,A)\mathrm{Q}_j(s')B(s')A(s)}{\sum_s\sum_{s'} P(s|\varepsilon_i)P(s'|s,A)\mathrm{Q}_j(s')}, \end{aligned} \tag{13}$$

where the observable $\mathrm{Q}_j(s)$, an *indicator function*, is such that

$$\mathrm{Q}_j(s) = \begin{cases} 1 \text{ if } E(s) = \varepsilon_j, \\ 0 \text{ otherwise.} \end{cases} \tag{14}$$

In terms of fragile expectations, C_{ij} can be written as $C_{ij} = \big\langle (B\mathrm{Q}_j)*A\big\rangle_{\varepsilon_i}/\big\langle \mathrm{Q}_j * \mathrm{I}_A\big\rangle_{\varepsilon_i}$ where I_A is an observable with the same transition probabilities as A, that is, $P(s'|s,A)$, but $\mathrm{I}_A(s) = 1\ \forall s$. The diagonal elements are simply $C_{ii} = \big\langle B*A\big\rangle_{\varepsilon_i}$.

CONNECTION WITH QUANTUM THEORY

We can formulate this theory using a complex Hilbert space just as in Von Neumann's formulation of quantum theory [7]. For this we consider an arbitrary orthonormal basis set $|n\rangle$ (n=1,...,N) with $\langle i|j\rangle = \delta(i,j)$ and define the density operator [8] $\hat{\rho}$ as

$$\hat{\rho} := \sum_{i=1}^{N}\sum_{j=1}^{N} \rho_{ij}\,|i\rangle\,\langle j|\,, \tag{15}$$

with ρ_{ij} complex numbers. Imposing that $\hat{\rho}$ is Hermitian, we see that the diagonal elements ρ_{ii} must be real and $\rho_{ij} = \overline{\rho_{ji}}$. It is always possible to make a choice of such complex matrix elements ρ_{ij} so that they are proportional to the elements p_{ij}, these are given by

$$\rho_{ij} = \langle\varepsilon_i|\,\hat{\rho}\,|\varepsilon_j\rangle = \frac{1}{2}\Big([p_{ij}+p_{ji}] + \boldsymbol{i}[p_{ij}-p_{ji}]\Big). \tag{16}$$

In the same way, we can write the measurement operator $\hat{C} = \hat{B}\hat{A}$, which must also be Hermitian, as

$$\hat{C} := \sum_{i=1}^{N}\sum_{j=1}^{N} c_{ij} \left|i\right\rangle \left\langle j\right|, \tag{17}$$

with matrix elements

$$c_{ij} = \langle \varepsilon_i | \hat{C} \left| \varepsilon_j \right\rangle = \frac{1}{2}\Big([C_{ij} + C_{ji}] + \boldsymbol{i}[C_{ij} - C_{ji}] \Big). \tag{18}$$

Please note that the choice of a complex Hilbert space over a real Hilbert space is unavoidable whenever $C_{ij} \neq C_{ji}$, because there will be complex matrix elements in the representation of the operator $\hat{C}$. This occurs whenever the measurement of A in $\big\langle B * A \big\rangle_I$ does not commute with a measurement of E. Of course, it is possible to choose the basis vectors $|i\rangle$ such that $\hat{C}$ is diagonal and has only real matrix elements, but this would not be possible in general for two different operators simultaneously. Under the choices in eq. (16) and eq. (18), and after some algebra, the trace of $\hat{\rho} \cdot \hat{C}$ is given by

$$\begin{aligned}\mathrm{Tr}\big(\hat{\rho} \cdot \hat{C}\big) &= \sum_{i,j} \rho_{ij} c_{ji} = \sum_i \Big\{ p_{ii} C_{ii} + \frac{1}{4}\sum_{j\neq i}\Big[(p_{ij} + p_{ji})(C_{ij} + C_{ji}) + (p_{ij} - p_{ji})(C_{ij} - C_{ji}) \Big] \Big\} \\ &= \sum_i \Big\{ p_{ii} C_{ii} + \frac{1}{2}\sum_{j\neq i}\Big[p_{ij} C_{ij} + p_{ji} C_{ji} \Big] \Big\} = \sum_{i=1}^{N}\sum_{j=1}^{N} p_{ij} C_{ij} = \Big\langle C \Big\rangle_I.\end{aligned} \tag{19}$$

reproducing the trace rule of quantum theory. Therefore, for any finite-dimensional fragile problem, we have shown that:

(1) The state of knowledge I can be represented by a density operator $\hat{\rho}$.

(2) Any observable given by a non-commutative product $B * A$ can be represented by a Hermitian operator $\hat{C}$.

(3) Fragile expectations such as $\big\langle B * A \big\rangle_I$ are given by the usual trace rule of quantum theory,

$$\big\langle B * A \big\rangle_I = \mathrm{Tr}\big(\hat{\rho} \cdot \big[\hat{B} \cdot \hat{A}\big]\big). \tag{20}$$

The choice of basis $|1\rangle, \ldots, |N\rangle$ remains arbitrary, as is the choice of the observable E. However, these two are not independent, but actually linked. There is a particularly interesting choice of E for which the quantum formalism gets simplified, namely E such that $P(\varepsilon'_j | \varepsilon_i, I) = \delta(i,j)$. If this is the case then $p_{ij} = P(\varepsilon'_j, \varepsilon_i | I) = P(\varepsilon_i | I) P(\varepsilon'_j | \varepsilon_i, I) = \delta(i,j) P(\varepsilon_i | I)$ and we see that this is the representation in which the density operator $\hat{\rho}$ is diagonal and has unit trace,

$$\mathrm{Tr}\ (\hat{\rho}) = \sum_{i=1}^{N} p_{ii} = \sum_{i=1}^{N} P(\varepsilon_i | I) = 1.$$

The eigenstates of $\hat{\rho}$ must then be interpreted as pure quantum states with definite values of E. The density operator $\hat{\rho}$ and the measurement operator $\hat{C}$ are then

$$\hat{\rho} = \sum_{i=1}^{N} P(\varepsilon_i | I) \left|\varepsilon_i\right\rangle \left\langle \varepsilon_i\right|, \tag{21}$$

$$\hat{C} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} \Big([C_{ij} + C_{ji}] + \boldsymbol{i}[C_{ij} - C_{ji}] \Big) \left|\varepsilon_i\right\rangle \left\langle \varepsilon_j\right|. \tag{22}$$

In this basis it is easier to verify eq. (20), as we have

$$\mathrm{Tr}\big(\hat{\rho} \cdot \hat{C}\big) = \sum_{i=1}^{N} \langle \varepsilon_i | \sum_{j=1}^{N} P(\varepsilon_j | I) \left|\varepsilon_j\right\rangle \left\langle \varepsilon_j\right| \hat{C} \left|\varepsilon_i\right\rangle = \sum_{i=1}^{N} P(\varepsilon_i | I) \left\langle \varepsilon_i\right| \hat{C} \left|\varepsilon_i\right\rangle. \tag{23}$$

The matrix element $\langle \varepsilon_i | \hat{C} | \varepsilon_i \rangle$ is in fact equal to $\big\langle B(s')A(s) \big\rangle_{\varepsilon_i, \varepsilon'_i, A}$, the fragile expectation of $B * A$ where we have constrained the initial and final values of E to be equal to ε_i. Replacing this into eq. (23), we verify that

$$\mathrm{Tr}\big(\hat{\rho} \cdot \hat{C}\big) = \sum_{i=1}^{N} P(\varepsilon_i | I) \Big\langle B(s')A(s) \Big\rangle_{\varepsilon_i, \varepsilon'_i, A} = \Big\langle B(s')A(s) \Big\rangle_{I,A} = \Big\langle B * A \Big\rangle_I. \tag{24}$$

CONCLUDING REMARKS

We have developed a framework for non-commutative Bayesian expectations, in which measurements of observables have side effects. By expressing the formalism in a way that is independent of the internal states of the system, we arrive at a formulation based on operators in a complex Hilbert space which is remarkably similar to quantum theory. This result supports the view that quantum theory is a particular framework of probabilistic inference under requirements which are absent from "classical" inference, in this case the side effects produced by the act of gathering information. Of course, the idea of quantum theory being some form of Bayesian inference is not new: it has been most visibly put forward by the *Quantum Bayesianism* (QBism for short) interpretation by Fuchs et al. [9, 10, 11], but also other frameworks that connect the mathematics of density operators with Bayesian inference [12].

Although we are primarily interested in the validation of a framework for doing non-commutative inference in the contexts of invasive measurements of classical systems and even in the context of psychology or cognition, the present ideas may provide elements for an eventual reconstruction of *quantum physical laws* from Bayesian probability, in an abstract manner independent of representations. This is in line with the idea that models, in particular probabilistic models, are not intrinsic to phenomena, but instead, that the peculiar mathematical form of our laws of Nature, as seen, for instance, in physics, are mostly a manifestation of our optimal protocols for doing reasoning under incomplete information (i.e. inference) about Nature, information that we can acquire from observation. As these phenomena will fall into certain patterns, this might point as to why it is possible to apply models and frameworks traditionally considered as exclusively part of the domain of physics throughout different fields of Science.

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